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TREATMENTS OF OVER-RESTRAINED BOUNDARIES FOR DOUBLY CONNECTED PLATES OF ARBITRARY SHAPE IN VIBRATION ANALYSIS

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Abstract—The problem of the free flexural vibrations of plates of doubly connected arbitrary shape is investigated. A hybrid energy approach which combines both the advantages of the Rayleigh–Ritz method and the Lagrange multiplier method is developed for analysing the aforementioned plate problems. The Rayleigh–Ritz method with a set of *pb*-2 shape functions is used to formulate the plates with classical boundary conditions as a single continuum element, while the Lagrange multipliers are imposed at discrete points in the inner plate boundaries, for instance, when the over-restrained boundary has occurred. The admissible *pb*-2 shape function consists of the product of a two-dimensional polynomial and a basic function. The basic function is defined by the product of the equations of the prescribed piecewise continuous boundary shape each raised to the power of 0, 1 or 2, corresponding respectively to a free, simply-supported or clamped edge. This set of functions automatically satisfies all the kinematic boundary conditions of the plate at the outset. Numerical results for several examples of doubly connected plates of arbitrary shape are presented to demonstrate the applicability and accuracy of the present method.

NOTATION

A	area of the plate
c_i	coefficients used in deflection function
D	flexural rigidity $= Eh^3/12\,(1-\nu^2)$
$\mathrm{d}A$	elemental area of plate $= \mathrm{d}x\,\mathrm{d}y$
F	total energy functional
F^*	augmented energy functional
$f(x, y)$	generating function
h	plate thickness
T	kinetic energy
U	strain energy
$w(x, y)$	deflection function
x, y	Cartesian coordinates
ρ	density per unit volume of plate
ω	natural frequency
λ	frequency parameters $= \rho h\omega^2 a^4/D$
Λ	Lagrangian multiplier
$\Gamma(x,y)$	boundary expression
$\Phi_1(x, y)$	basic function.

1. INTRODUCTION

The pioneering work of Chladni (1802) has created tremendous research interest in plate vibrations for almost two centuries. A vast amount of research work has been reported on this topic (Leissa, 1969, 1977, 1981, 1987; Liew, 1990). For some simple plate shapes and boundary conditions, exact solutions to the problems by analytical methods are possible. Apart from the exact analytical methods, approximate methods must be employed for the solution of plates of arbitrary shape and with complicated boundary conditions.

Over a long period of research and development, many researchers have realized that the Rayleigh–Ritz method (1909) is one of the most powerful numerical tools for vibration analysis. The method has been widely used to solve plate vibration problems due to its simplicity in numerical implementation. The accuracy and rate of convergence of this method are highly dependent on the choice of the admissible functions in the deflection series. Although trigonometric and hyperbolic series (Leissa, 1969; Gorman, 1982) are the effective Ritz functions for certain plate shapes, their application to plates of arbitrary shape may not be so convenient. It is often a tedious task to define functions that satisfy support conditions along the edges. One convenient strategy is to employ the *pb*-2 shape functions (Liew, 1992) which ensure satisfaction of the kinematic boundary conditions. These sets of shape functions work well for non-perforated plates of arbitrary shape in vibration analysis (Liew, 1990, 1992).

Apparently very limited research work has been reported in the open literature on the free vibration analysis of doubly connected plates. We can find several research papers for the vibration of annular plates (Vogel and Skinner, 1965; Ramaiah and Vijayakumar, 1972, 1973; Avalos and Laura, 1979; Gorman, 1982, 1983; Narita, 1984; Gelos *et al.*, 1985; Bianchi *et al.*, 1985; Kanaka Raju and Venkateswara, 1986). There have also been a couple of research papers published on rectangular plates with circular (Kumai, 1952; Takahasi, 1958; Hegarty and Ariman, 1975) or rectangular openings (Paramasivam, 1973; Rajamani and Prabhakaran, 1977a,b; Lam and Hung, 1990). For the vibration analysis of doubly connected plates of arbitrary shape, only two excellent papers can be found (Nagaya, 1981a, b). The method used by Nagaya involves the partitioning of the entire plate domain into small segments, and obtaining the general transformed expressions for the bending slope, bending moment and shearing force of a segment under the assumption that the angle between the normal to the segment and the reference axis is constant. The boundary conditions along the outer and inner edges are satisfied by means of the Fourier expansion collocation method. This method is obviously a very powerful approximate tools for doubly connected plate vibration analysis but it is quite difficult to be implemented numerically.

The aim of this paper is to present a simple and accurate method for the free vibration analysis of doubly connected plates of arbitrary shape as shown in Fig. 1. The entire plate domain is treated as a single continuum element in the solution procedure. The analysis is performed by using a set of *pb*-2 shape functions in the Rayleigh–Ritz method to formulate the governing eigenvalue equation. The *pb*-2 shape function consists of the product of a two-dimensional polynomial and a basic function. The basic function defined by the product of equations of the specified boundary shape each raised to the power of 0, 1 or 2 corresponds respectively to a free, simply-supported or clamped edge. The set of basic functions automatically satisfies the kinematic boundary conditions of the plate at the outset. For plate shapes with boundary equations passing through the internal plate domain which imposed the unwanted restrictions [imaginary lines of the boundary equations in Fig. 2(a)], some

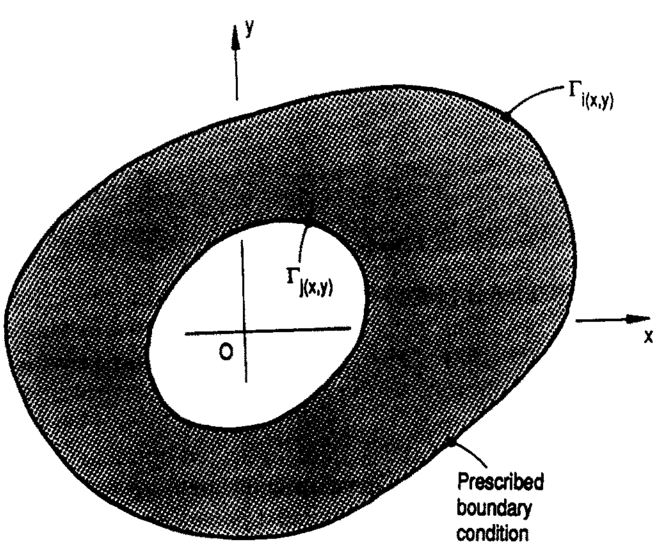


Fig. 1. Geometry of a doubly connected plate.

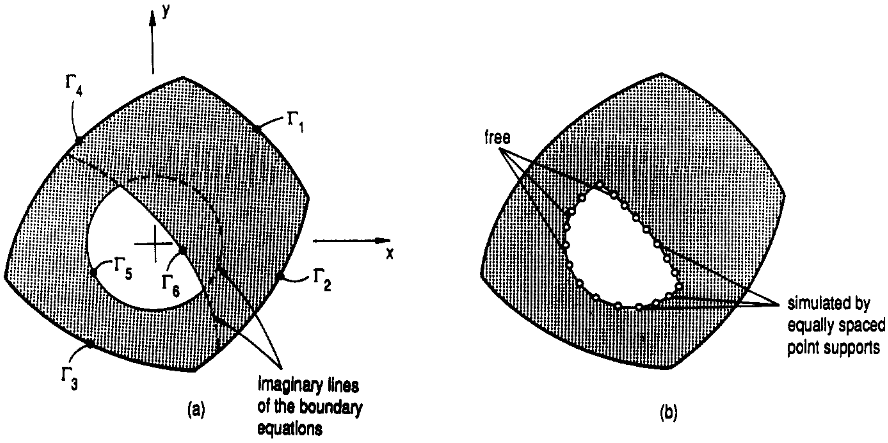


Fig. 2. (a) A plate with boundary equations passing through the internal plate domain, and (b) simulation of boundary conditions by a series of equally spaced point supports.

special treatments for the over-restrained edges are needed. First we have to make the entire edge free by inserting zero power to the boundary equation and a series of closely spaced simple or clamped point constraints are then used to simulate the prescribed boundary conditions [see Fig. 2(b)]. These constraints are imposed by making (a) the deflection equal to zero [$w(x, y) = 0$] for simply-supported points, and (b) the deflection and normal slope equal to zero [$w(x, y) = \partial w/\partial n = 0$] for clamped points.

A computer software code "SEPSA" has been recently developed by a research team in Nanyang Technological University (NTU) which is based on the above theories. This code is readily applicable to a wide class of doubly connected plate problems. In the present study, natural frequency parameters for several selected doubly connected plate problems are obtained to demonstrate the applicability and accuracy of the hybrid energy approach. The present results, where possible, are compared with the existing solutions from the open literature.

2. METHOD OF FORMULATION

Consider a flat, isotropic and elastic thin doubly connected plate of constant thickness h as shown in Fig. 1. The strain energy, U, of the considered plate is given by

$$U = \frac{D}{2}\int_A \left\{\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left[\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right]\right\} \mathrm{d}A, \tag{1}$$

where D = flexural rigidity of the plate; ν = Poisson's ratio; $w(x, y)$ = the deflection of the middle plane of the plate perpendicular to the x–y plane; and A = area of the plate.

For free vibration analysis, the kinetic energy of the plate is given by

$$T = \tfrac{1}{2}\rho h\omega^2 \int_A w^2\,\mathrm{d}A, \tag{2}$$

where ρ = the mass density per unit volume of the plate and ω = angular frequency of vibration.

The expression for the total energy functional, F, of the plate is given by

$$F = U - T. \tag{3}$$

The deflection surface $w(x, y)$ for the plate may be expressed as

$$w(x,y) = \sum_{i=1}^{m} c_i \Phi_i(x,y), \tag{4}$$

where c_i = the unknown coefficients to be minimized in the Rayleigh–Ritz procedure.

The *pb*-2 Ritz function $\Phi_i(x, y)$ is given by

$$\Phi_i(x,y) = f_i(x,y)\Phi_1(x,y), \tag{5}$$

where Φ_1 = the basic function which formed as the product of the boundary expressions and f_i = the generating function.

The basic function Φ_1 for the plate is given by

$$\Phi_1(x,y) = \prod_{j=1}^{n_s} [\Gamma_j(x,y)]^{\Omega_j}, \tag{6}$$

where Γ_j = the boundary expression of the jth supporting edge, n_s = the number of supporting edges and Ω_j depending on the supporting edge condition, takes on

$$\begin{cases} \Omega_j = 0 & \text{for } j\text{th free edge,} \\ \Omega_j = 1 & \text{for } j\text{th simply-supported edge,} \\ \Omega_j = 2 & \text{for } j\text{th clamped edge.} \end{cases} \tag{7}$$

The generating function $f_i(x, y)$ may be formed as follows:

$$f_i(x,y) = x^r y^s \cos^2\frac{\pi(i-r^2-1)}{2} + x^s y^r \sin^2\frac{\pi(i-r^2-1)}{2}, \tag{8}$$

where

$$r = \lceil \sqrt{i-1} \rceil, \quad s = \left[\frac{i-r^2-1}{2}\right]\cos^2\frac{\pi(i-r^2-1)}{2} + \left[\frac{i-r^2-2}{2}\right]\sin^2\frac{\pi(i-r^2-1)}{2}, \tag{9}$$

in which $\lceil\ \rceil$ = the greatest integer function, for example $\lceil\sqrt{3}\rceil = 1$. The number of polynomials can be generated up to m terms by eqn (8), rather than the conventional procedure of generating the polynomial terms in a degree set.

Note that Φ_1 ensures the Ritz functions satisfy the kinematic boundary conditions for plates with classical boundary conditions. However, for plates with over-restrained conditions at the internal boundaries [see Fig. 2(b)], a series of closely spaced simply-supported or clamped points are introduced to simulate the prescribed support conditions. If the plate has N point supports either along its edges or internally, the following constraints must be imposed:

$$w_i = 0, \quad i = 1,2,\ldots,N_s \tag{10}$$

for simple point supports, and

$$w_i = \frac{\partial w_i}{\partial n} = 0, \quad i = 1,2,\ldots,N_c \tag{11}$$

for clamped point supports where w_i = the deflection at the ith point support; N_s = number of simple point supports; N_c = number of clamped point supports; and n is the normal direction to the clamped line. These constraints may be satisfied by augmenting the unified functional, F, of eqn (3) to

$$F^* = U - T + \sum_{p=1}^{N_s} \Lambda_p w_p(x_s, y_s) + \sum_{q=1}^{N_c} \Lambda_q^* w_q(x_c, y_c) + \sum_{r=N_c+1}^{2N_c} \Lambda_r^* \frac{\partial w_r(x_c, y_c)}{\partial n}, \tag{12}$$

where Λ_p, Λ_q^*, $\Lambda_r^*$are Lagrangian multipliers, (x_s, y_s), and (x_c, y_c) are the position coordinates of simple and clamped point supports, respectively.

The minimization of the augmented functional, F^*, leads to the governing eigenvalue equation

$$\left(\begin{bmatrix} K & L^T \\ L & 0 \end{bmatrix} - \lambda \begin{bmatrix} M & 0 \\ 0 & 0 \end{bmatrix}\right) \begin{Bmatrix} c \\ \Lambda \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \tag{13}$$

where $\lambda = \rho h\omega^2 a^4/D$, $\{c\} = \{c_1, c_2, \ldots, c_m\}^{\mathrm{T}}$, $\{\Lambda\} = \{\Lambda_1, \Lambda_2, \ldots, \Lambda_{N_s}, \Lambda_1^*, \Lambda_2^*, \ldots, \Lambda_{2N_c}^*\}^{\mathrm{T}}$, and the elements in the matrices are

$$K_{ij} = R_{ij}^{2020} + R_{ij}^{0202} + \nu(R_{ij}^{0220} + R_{ij}^{2002}) + 2(1-\nu)R_{ij}^{1111}, \tag{14}$$

$$L_{pi} = \Phi_i(x_p, y_p), \quad p = 1, 2, \ldots, N_s, \tag{15}$$

$$L_{qi} = \Phi_i(x_q, y_q), \quad q = 1, 2, \ldots, N_c, \tag{16}$$

$$L_{ri} = \frac{\partial \Phi_i(x_r, y_r),}{\partial n}, \quad r = 1, 2, \ldots, N_c, \tag{17}$$

$$M_{ij} = R_{ij}^{0000} \tag{18}$$

and

$$R_{ij}^{tuvw} = \int_A \left[\frac{\partial^{t+u}\Phi_i(x, y)}{\partial x^t\, \partial y^u}\right]\left[\frac{\partial^{v+w}\Phi_j(x, y)}{\partial x^v\, \partial y^w}\right] \mathrm{d}A \tag{19}$$

in which $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, m$. For vibration analysis, the eigenvalues are obtained by solving the set of homogeneous equations (13).

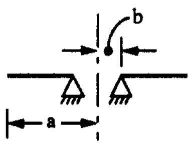
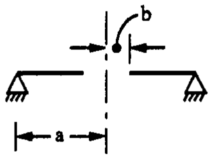
3. RESULTS AND DISCUSSION

In order to demonstrate the applicability and accuracy of the present approach, numerical results for several doubly connected plates of arbitrary shape having different inner and outer boundary conditions are determined using the recently developed computer software code "SEPSA". Poisson's ratio when needed is taken to be 0.3. The numerical results obtained for the plate problems are expressed in terms of the non-dimensional frequency parameters $(\rho h\omega^2 a^4/D)^{1/4}$. The results presented for all the examples are obtained using $m = 100$. These 100 terms employed in the deflection series can provide quite acceptable convergence.

The computer software code "SEPSA" is first applied to solve a series of annular plate problems with eight possible combinations of the inner and outer boundary conditions. The symbol S-C represents an annular plate with the inner edge simply-supported and the outer edge clamped. The eight cases considered in the first set of problems are the annular plates with S-F, F-S, C-F, F-C, S-S, C-C, S-C and C-S boundary conditions, respectively.

The non-dimensional frequency parameters of the eight annular plates with different

Table 1. Comparison of frequency parameters $(\rho h\omega^2 a^4/D)^{1/4}$ for annular plates with a combination of free and simply-supported edges.

Boundary conditions	b/a	Source of results	Mode sequence number 1	2	3	4	5	6
	0.3	Present	1.840	1.853	1.853	2.467	2.467	3.562
		Gorman	1.837	1.850	1.850	2.466	2.466	3.551
	0.5	Present	2.031	2.209	2.209	2.834	2.834	3.748
		Gorman	2.030	2.205	2.205	2.826	2.826	3.746
	0.7	Present	2.487	2.890	2.890	3.668	3.668	4.528
		Gorman	2.487	2.890	2.890	3.664	3.664	4.523
	0.3	Present	2.162	3.594	3.594	4.917	4.917	6.097
		Gorman	2.159	3.580	3.580	4.911	4.911	6.086
	0.5	Present	2.253	3.409	3.409	4.734	4.734	5.987
		Gorman	2.253	3.407	3.407	4.728	4.728	5.970
	0.7	Present	2.632	3.648	3.648	4.932	4.932	6.092
		Gorman	2.632	3.648	3.648	4.932	4.932	6.091

combinations of boundary conditions for various b/a ratios are presented in Tables 1–4, respectively. The first six frequency parameters shown in the tables are presented in an increasing sequence. For the doubly connected annular plate problems, Gorman (1982) has presented a comprehensive set of reasonably accurate vibration frequencies for all the eight cases considered here. The finite element method was used in his studies. In the tables, the finite element solutions (Gorman, 1982) are also included for comparison purposes. From these tables, it can be observed that very close agreements have been obtained for all eight annular plates considered.

The second problem considered is a fully clamped square plate with an internal clamped square cut-out. In this example if the piecewise boundary equations are used to form the basic function, it causes an unnecessary restraint on the internal boundary equations that

Table 2. Comparison of frequency parameters $(\rho h\omega^2 a^4/D)^{1/4}$ for annular plates with a combination of free and clamped edges.

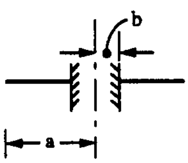
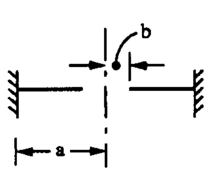
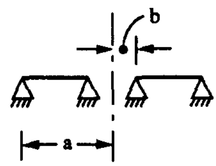
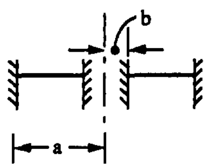
Boundary conditions	b/a	Source of results	Mode sequence number 1	2	3	4	5	6
	0.3	Present	2.639	2.781	2.781	2.877	2.877	4.078
		Gorman	2.637	2.797	2.797	2.873	2.873	4.069
	0.5	Present	3.691	3.729	3.729	3.806	3.806	4.461
		Gorman	3.691	3.728	3.728	3.799	3.799	4.454
	0.7	Present	6.086	6.105	6.105	6.201	6.201	6.464
		Gorman	6.086	6.105	6.105	6.199	6.199	6.460
	0.3	Present	3.382	4.423	4.423	5.710	5.710	7.009
		Gorman	3.380	4.420	4.420	5.709	5.709	7.005
	0.5	Present	4.209	4.691	4.691	5.667	5.667	6.771
		Gorman	4.209	4.692	4.692	5.667	5.667	6.769
	0.7	Present	6.568	6.732	6.732	7.182	7.182	7.829
		Gorman	6.568	6.733	6.733	7.182	7.182	7.829

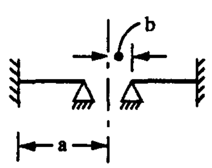
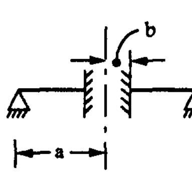
Table 3. Comparison of frequency parameters $(\rho h\omega^2 a^4/D)^{1/4}$ for annular plates with either simply-supported or clamped edges.

Boundary conditions	b/a	Source of results	Mode sequence number					
			1	2	3	4	5	6
	0.3	Present	4.592	4.831	4.831	5.506	5.506	6.481
		Gorman	4.592	4.829	4.829	5.502	5.502	6.474
	0.5	Present	6.328	6.466	6.466	6.864	6.864	7.487
		Gorman	6.328	6.465	6.465	6.862	6.862	7.481
	0.7	Present	10.49	10.56	10.56	10.75	10.75	11.06
		Gorman	10.49	10.55	10.55	10.75	10.75	11.07
	0.3	Present	6.734	6.834	6.834	7.159	7.159	7.757
		Gorman	6.734	6.830	6.830	7.152	7.152	7.749
	0.5	Present	9.448	9.501	9.501	9.663	9.663	9.946
		Gorman	9.448	9.499	9.499	9.660	9.660	9.946
	0.7	Present	15.76	15.80	15.80	15.85	15.85	15.98
		Gorman	15.76	15.79	15.79	15.86	15.86	15.98

extend into the plate domain. To overcome this difficulty the internal edges are made free and the clamped edge conditions are approximated by a series of clamped point supports. For this example, the clamped condition for each internal edge is approximated by imposing a series of eight equally spaced clamped point supports (see Fig. 3). Using this modified approach, the first five frequency parameters obtained for this example are presented in Fig. 3 together with the numerical and experimental values of Nagaya (1981b). As shown in the figure, it can be seen that the present results agree well with both the numerical and experimental values of Nagaya.

The third problem considered is a fully clamped square plate with an internal clamped elliptical hole. For this example, if detailed study is required, many parameters can be varied such as the $2b'/a$ and a'/b' ratios. However, the doubly connected plate with only a

Table 4. Comparison of frequency parameters $(\rho h\omega^2 a^4/D)^{1/4}$ for annular plates with a combination of simply-supported and clamped edges.

Boundary conditions	b/a	Source of results	Mode sequence number					
			1	2	3	4	5	6
	0.3	Present	5.811	5.994	5.994	6.544	6.544	7.401
		Gorman	5.811	5.992	5.992	6.537	6.537	7.390
	0.5	Present	7.999	8.094	8.094	8.377	8.377	8.844
		Gorman	7.998	8.092	8.092	8.375	8.375	8.842
	0.7	Present	13.21	13.25	13.25	13.37	13.37	13.58
		Gorman	13.21	13.25	13.25	13.37	13.37	13.58
	0.3	Present	5.475	5.611	5.611	6.026	6.026	6.758
		Gorman	5.475	5.604	5.604	6.021	6.021	6.743
	0.5	Present	7.734	7.811	7.811	8.041	8.041	8.434
		Gorman	7.734	7.810	7.810	8.039	8.039	8.433
	0.7	Present	12.98	13.02	13.02	13.13	13.13	13.32
		Gorman	12.98	13.02	13.02	13.13	13.13	13.32

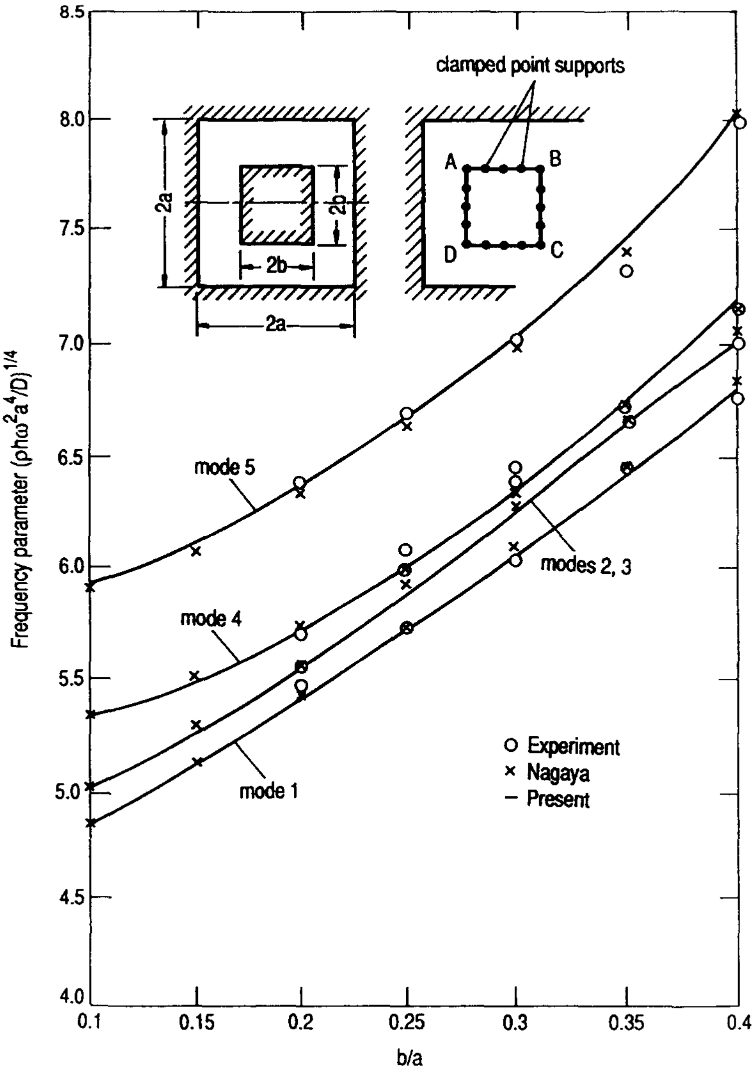


Fig. 3. Comparison between theoretical and experimental frequency parameters $(\rho h\omega^2 a^4/D)^{1/4}$ for a clamped square plate with a clamped internal square hole.

fixed ratio $2b'/a$ of 0.3 is investigated here. The results obtained for this example with various a'/b' ratios are presented in Fig. 4. The experimental and numerical values of Nagaya (1981b) are plotted together with the present results in the same figure in order to make a clear comparison. From this figure, we can clearly see that the present results agree well with those experimental and numerical values.

The last problem considered is a fully clamped circular plate with an internal clamped elliptical hole. The detailed geometry and dimensions of this doubly connected irregular annular plate are shown in Fig. 5. For this example, only a fixed ratio b'/a of 0.3 is considered. Of course, if more results are needed, the software code "SEPSA" can be used to generate solutions for different combinations of parameters. The first five frequency parameters are presented graphically in Fig. 5. These results are obtained for the doubly connected plate with a fixed ratio b'/a of 0.3 but with different values of the a'/b' ratio. Again it can be observed that the present results are comparable to the published experimental and numerical values of Nagaya (1981b).

4. CONCLUSIONS

This paper presents a hybrid energy approach for the free vibration analysis of doubly connected plates of arbitrary shape. As discussed in the main text, this method combines

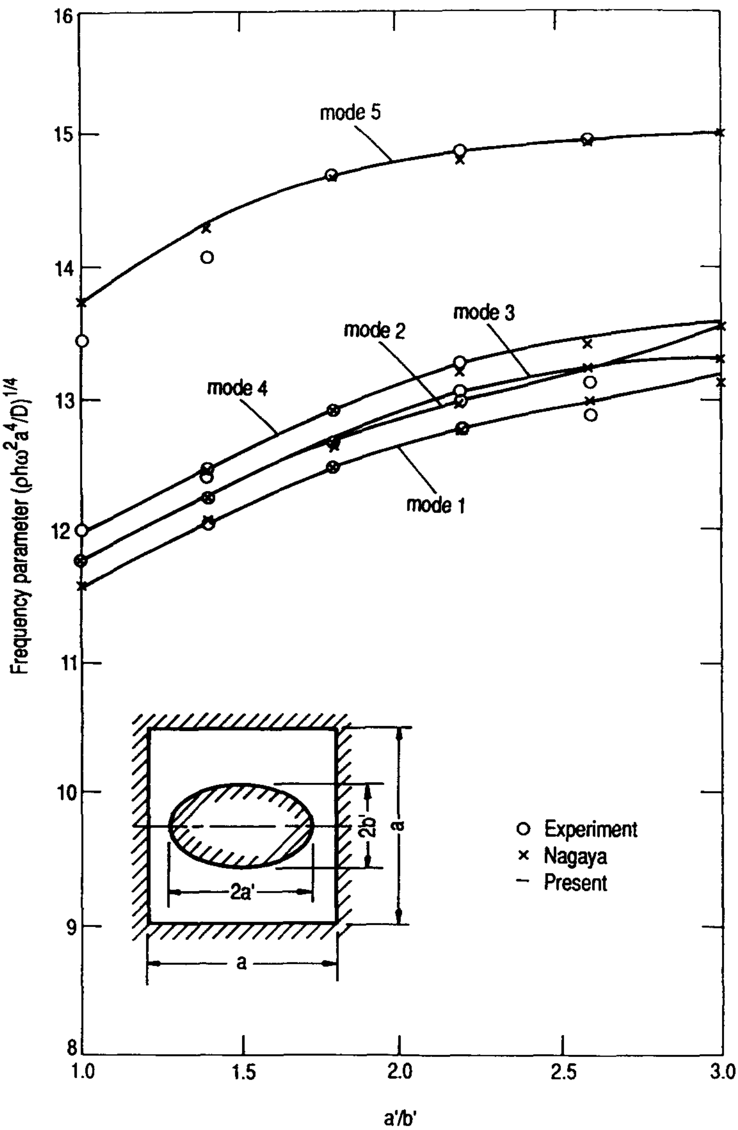


Fig. 4. Comparison between theoretical and experimental frequency parameters $(\rho h\omega^2 a^4/D)^{1/4}$ for a clamped square plate with a clamped internal elliptical hole $(2b'/a = 0.30)$.

both the advantages of the Rayleigh–Ritz method and the Lagrange multiplier method. A software code "SEPSA" has been recently developed based on the proposed theories.

The hybrid energy approach, to the author's knowledge, appears to be the first successfully applied to this class of plate vibration problems. In this paper, some treatments have been suggested to overcome the difficulties encountered in over-restrained edges. The present method is very different from the discretization methods such as the finite difference, finite element or finite strip method because only one single element is used for the whole solution procedure. The author believes that the present method should provide more accurate solutions to this class of problems when a sufficient number of terms has been used in the deflection series.

Frequency parameters for several plate problems have been obtained and compared with the established solutions from the open literature. Close agreements are obtained for all the examples, therefore it verifies the accuracy and applicability of the present method. Although only a few examples are presented here, the present method is readily applicable to solving a wide range of doubly connected plate problems of arbitrary shape.

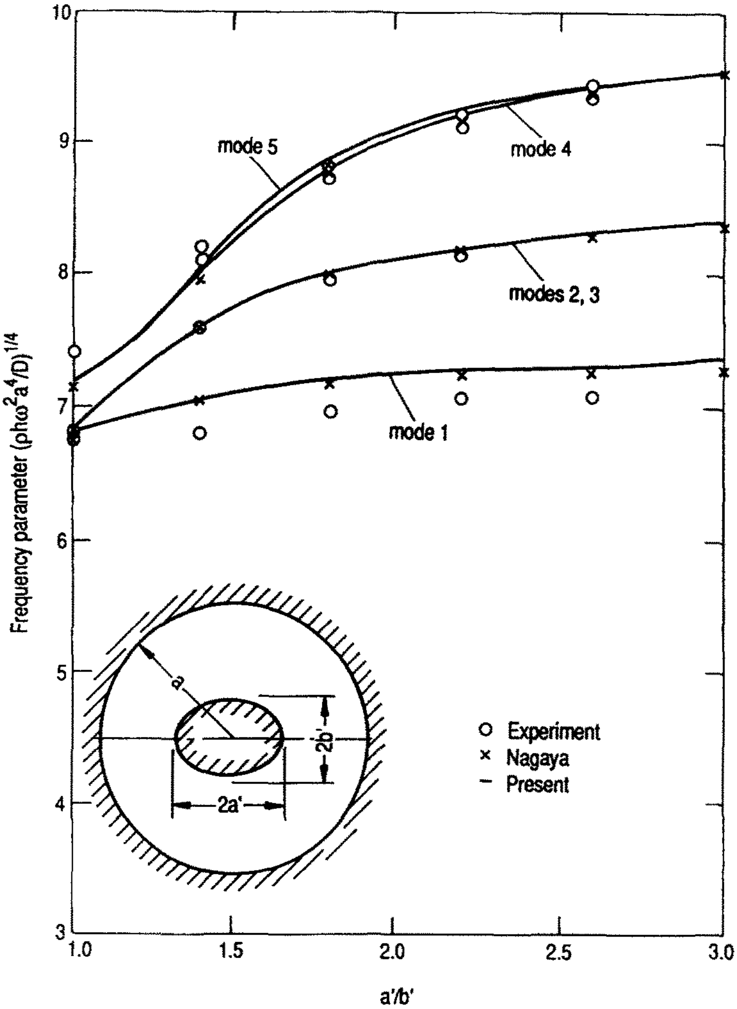


Fig. 5. Comparison between theoretical and experimental frequency parameters $(\rho h\omega^2 a^4/D)^{1/4}$ for a clamped circular plate with a clamped internal elliptical hole ($b'/a = 0.30$).

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